

EXERCISE 11B.6.2 - EXTENSION

Note: To make these solutions easier to follow, we have abbreviated “vertex of degree n ” as D_n . So for example, if we refer to D_2 , it means “vertex of degree 2”.

3 a

Degrees	Form	Total Trees
1, 1		1

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b

Degrees	Form	Permutations	Total Trees
2, 1, 1		3 ways to choose D_2 .	3

c

Degrees	Form	Permutations	Total Trees
2, 2, 1, 1		4! ways to choose the order of vertices, $\div 2$ because the trees are the same in reverse.	12
			4
			16

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Degrees	Form	Permutations	Total Trees
1, 1		$\times 4$ ways to choose D_1 that is adjacent to D_2 .	60
4, 1, 1, 1, 1		5 ways to choose D_4 .	60
			125

e

Degrees	Form	Permutations	Total Trees
2, 2, 2, 2, 1, 1		6! ways to choose the order of vertices, $\div 2$ because the trees are the same in reverse.	360
3, 2, 2, 1, 1, 1		6! ways to choose the order of vertices, $\div 2$ since 2 D_1 are interchangeable.	360
3, 2, 2, 1, 1, 1		6! ways to choose the order of vertices, $\div 2$ since 2 D_2 are interchangeable.	360
3, 3, 1, 1, 1, 1		6! ways to choose the order of vertices, $\div 2$ since 2 D_3 are interchangeable, $\div 2$ for the first lot of 2 interchangeable D_1 , $\div 2$ for the second lot of 2 interchangeable D_1 .	90
4, 2, 1, 1, 1, 1		6! ways to choose the order of vertices, $\div 3!$ since 3 D_1 are interchangeable.	120
5, 1, 1, 1, 1, 1		6 ways to choose D_5 .	6
			1296

Notice that $1 = 2^0$, $3 = 3^1$, $16 = 4^2$, $125 = 5^3$, $1296 = 6^4$.

Hence for K_n , we postulate there are n^{n-2} trees.

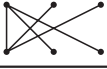
4 a

Degrees	Form	Total Trees
$\{1\}, \{1\}$		1

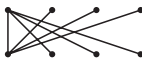
b

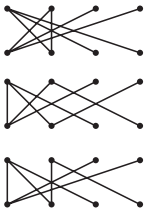
Degrees	Form	Total Trees
$\{2, 1\}, \{2, 1\}$	   	4

c

Degrees	Form	Permutations	Total Trees
$\{3, 1, 1\}, \{3, 1, 1\}$		3 ways to choose D_3 on top, $\times$ 3 ways to choose D_3 on bottom.	9
$\{3, 1, 1\}, \{2, 2, 1\}$		3 ways to choose D_3 on top, $\times$ 3 ways to choose D_1 on bottom, $\times$ 2 ways to choose how D_2 on bottom connects to D_1 on top.	18
$\{2, 2, 1\}, \{3, 1, 1\}$		symmetric to case $\{3, 1, 1\}, \{2, 2, 1\}$	18
$\{2, 2, 1\}, \{2, 2, 1\}$		3 ways to choose D_1 on top, $\times$ 3 ways to choose D_1 on bottom, $\times$ 2 ways to choose which vertex D_1 on top connects to, $\times$ 2 ways to choose which vertex D_1 on bottom connects to.	36
			81

d

Degrees	Form	Permutations	Total Trees
$\{4, 1, 1, 1\}, \{4, 1, 1, 1\}$		4 ways to choose D_4 on top, $\times$ 4 ways to choose D_4 on bottom.	16
$\{4, 1, 1, 1\}, \{3, 2, 1, 1\}$		4 ways to choose D_4 on top, $\times$ 4 ways to choose D_3 on bottom, $\times$ 3 ways to choose D_2 on bottom, $\times$ 2 ways to choose which D_1 the D_2 connects to.	144
$\{4, 1, 1, 1\}, \{2, 2, 2, 1\}$		4 ways to choose D_4 on top, $\times$ 4 ways to choose D_1 on bottom, $\times$ 3 ways to choose which D_1 the first D_2 connects to, $\times$ 2 ways to choose which D_1 the second D_2 connects to.	96
$\{3, 2, 1, 1\}, \{4, 1, 1, 1\}$		symmetric to case $\{4, 1, 1, 1\}, \{3, 2, 1, 1\}$	144

$\{3, 2, 1, 1\},$ $\{3, 2, 1, 1\}$		4 ways to choose D_3 on top, $\times 3$ ways to choose D_2 on top, $\times 4$ ways to choose D_3 on bottom, $\times 3$ ways to choose D_2 on bottom, $\times (1 \text{ way if the } D_3 \text{ aren't adjacent}$ $+ 4 \text{ ways if the } D_3 \text{ are adjacent and}$ $\text{both } D_3 \text{ also connect to } D_2,$ $+ 4 \text{ ways if the } D_3 \text{ are adjacent and}$ $\text{exactly one } D_3 \text{ also connects to } D_2$ $(\text{this is 2 for which } D_3, \times 2 \text{ for which}$ $D_1 \text{ the other } D_3 \text{ connects to}))$	1296
$\{3, 2, 1, 1\},$ $\{2, 2, 2, 1\}$		4 ways to choose D_3 on top, $\times 3$ ways to choose D_2 on top, $\times 4$ ways to choose D_1 on bottom. So, there are 48 times all of the following: - If D_3 is adjacent to D_1, we have: 3 choices for which D_2 the D_3 does <i>not</i> connect to, $\times 2$ choices for the connections of the D_2 on top (it <i>must</i> connect to the D_2 on the bottom that the D_3 does not connect to, then has 2 options for the other connection), $\times 2$ choices for which way the D_1 are connected. - If D_3 is <i>not</i> adjacent to D_1, we have: 3 choices for the connections of the D_2 on top (it <i>must</i> connect to the D_1 on the bottom, then has 3 options for the other connection), $\times 2$ choices for which way the D_1 are connected, i.e., we have $48 \times (12 + 6)$ ways in total.	864
$\{2, 2, 2, 1\},$ $\{4, 1, 1, 1\}$		symmetric to case $\{4, 1, 1, 1\}, \{2, 2, 2, 1\}$	96
$\{2, 2, 2, 1\},$ $\{3, 2, 1, 1\}$		symmetric to case $\{3, 2, 1, 1\}, \{2, 2, 2, 1\}$	864
$\{2, 2, 2, 1\},$ $\{2, 2, 2, 1\}$		4 ways to choose D_1 on top, $\times 4$ ways to choose D_1 on bottom, $\times 3$ ways to choose vertex on top (call this A) to connect to D_1 on bottom, $\times 3$ ways to choose vertex on bottom (call this B) to connect to D_1 on top, $\times 2$ ways to choose other vertex adjacent to A , $\times 2$ ways to choose other vertex adjacent to B .	576
4096			

Notice that $1 = 1^0$, $4 = 2^2$, $81 = 3^4$, $4096 = 4^6$.

Hence for K_n , we postulate there are n^{2n-2} trees.

5 For $K_{2,1}$ we have only 1 tree: 

By symmetry, $K_{1,2}$ must only have 1 tree also.

For $K_{3,2}$ we have:

Form	Permutations	Total Trees
	2 ways to choose D_3 on bottom, × 3 ways to choose which vertex D_1 on bottom connects to.	6
	3 ways to choose D_2 on top, × 2 ways to choose which vertex the first D_1 on top connects to.	6
		12

So, using question 4, we have these results:

$K_{1,1}$	$1 = 1^0 \times 1^0$
$K_{1,2}$	$1 = 1^1 \times 2^0$
$K_{2,1}$	$1 = 2^0 \times 1^1$
$K_{2,2}$	$4 = 2^1 \times 2^1$
$K_{3,2}$	$12 = 3^1 \times 2^2$
$K_{3,3}$	$81 = 3^2 \times 3^2$ etc.

Hence for $K_{m,n}$, we postulate there are $m^{n-1}n^{m-1}$ trees.